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ON THE RELATION OF THE OPERATOR $\partial/\partial s + \partial/\partial\tau$ TO EVOLUTION GOVERNED BY ACCRETIVE OPERATORS†

BY

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ABSTRACT

Existence theorems for the equation $du/dt + Au = f(t)$, where A is an accretive operator in a general Banach space X, are typically proved by showing that limits of solutions of discrete approximations to the equation exist. Here the estimates required to show this convergence are exhibited as special cases of estimates relating solutions of difference schemes for $\partial/\partial s + \partial/\partial\tau$ to exact solutions.

Introduction

Let X be a Banach space with the norm $\| \ \|$ and A be an accretive multivalued operator in X. Let $T > 0$, $f \in L^1(0, T: X)$, and consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} \text{(DE)} & \dfrac{du}{dt} + Au \ni f(t) \\ \text{(IC)} & u(0) = x_0. \end{cases}$$

This problem has been studied intensively in recent years, especially as regards the fundamental questions of existence and uniqueness of solutions.

If no additional restrictions are imposed on X, the basic method used to establish existence results has been to show, under various assumptions, the convergence of solutions of approximate difference schemes tending to (CP). The first general result of this kind was established in [4]. Extensions in several directions were obtained by numerous authors, e.g. [1], [8], [9], [10], [11].

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The uniqueness question is complicated by the fact that (DE) may not have any differentiable solutions even if there is a function u which is the limit of solutions of approximating difference schemes and which therefore ought to be the solution of (CP). One thus needs a way to interpret such a u as a solution of (DE). Toward this end two ideas have emerged. The first, already mentioned in [4] and developed further in [7], [10], [11], is essentially to define "solutions" of (DE) to be limits of solutions of approximating difference schemes. It must be proved that this notion is consistent with more classical ideas. The second idea, effectively developed by Benilan [1], [2], is to show that certain integral inequalities satisfied by limits of difference schemes are enough (together with Cauchy data) to uniquely determine them. Hence one can regard these inequalities as defining a notion of solution of the equation (DE). These two ideas are closely related and Benilan's uniqueness results provide an efficient way to prove that the notion "limits of solutions of difference schemes" is consistent with concepts of solutions of (DE) involving differentiability of u.

The current paper was partly motivated by two considerations. First, S. Parter pointed out to the authors a relationship between the convergence proof of [4] and a discrete approximation of the differential operator $\partial/\partial s + \partial/\partial\tau$. Secondly, in his interesting work [10] Takahashi showed how the arguments employed by Benilan in his uniqueness proof could be adapted to establish general convergence theorems. These arguments (called "Benilan's method") are not transparent; and, in the form used by Benilan and Takahashi, the domain of their applicability is not clear. Here we extract what seems to be at the core of these arguments in estimates that may be referred to in other situations. These estimates concern the degree of approximation by solutions of difference schemes to the exact solution of a boundary value problem involving the differential operator $\partial/\partial s + \partial/\partial\tau$. It is interesting that these estimates are not necessary for the proof of Benilan's uniqueness theorem. We also show how this result may be easily obtained from a simple argument again involving the differential operator $\partial/\partial s + \partial/\partial\tau$.

While this paper is nearly self-contained, it presumes some familiarity with the subject. An introduction is available in [3]. Section 1 contains the statement of the main convergence theorem for (CP) and some preliminary reductions. Section 2 contains the main estimates related to the operator $\partial/\partial s + \partial/\partial\tau$ and the proof of the theorem of Section 1. Section 3 contains a new proof of Benilan's uniqueness theorem. Rather than interrupt the presentation in Section 2 with numerous comments about extensions and special cases, we have chosen to collect these in a separate place, the final Section 4.

1. Preliminary reductions

Let $f \in L^1(0, T: X), x_0 \in X$ and $0 = t_0^n < \cdots < t_{N(n)}^n = T(n)$ be a partition of $[0, T(n)]$ for $n = 1, 2, \cdots$. Let $x_k^n, f_k^n \in X$ for $k = 0, 1, \cdots, N(n)$ and $n = 1, 2, \cdots$. Assume that

$$\frac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} + Ax_k^n \ni f_k^n \text{ for } k = 1, \cdots, N(n). \tag{1.1}$$

Denote by u_n and f_n the functions on $(0, T(n)]$ whose values on $(t_{k-1}^n, t_k^n]$ are x_k^n and f_k^n respectively. We also set $u_n(0) = x_0^n$. Provided that $\max_k (t_k^n - t_{k-1}^n) \to 0$, $x_0^n \to x_0$, and $f_n \to f$ (in some sense) we may hope that $u_n \to u$ where u is a solution of (CP). In fact one has:

THEOREM 1.2. *Let A be accretive, (1.1) hold, and $\bar{T} \geqq T(n) \geqq T > 0$ for large n. Let $f \in L^1(0, \bar{T}: X)$, and*

$$\lim_n \| f - f_n \|_{L^1(0,T(n):X)} = \lim_n \max_{1 \leqq k \leqq N(n)} (t_k^n - t_{k-1}^n) = 0.$$

If $\lim_n x_0^n = x_0 \in \overline{D(A)}$, then $u = \lim_n u_n$ exists uniformly on $[0, T]$ and u is continuous.

Theorem 1.2 will be proved in Section 2 as a corollary of the developments there. Assuming that $f = 0$ and an additional stability condition is satisfied, Takahashi [10] proved the convergence assertion of Theorem 1.2. Kobayashi [6] improved Takahashi's result by eliminating the stability assumption and obtaining more concrete estimates in a simpler way. Kobayashi's note came to the attention of the authors after most of the research in the current paper was complete. There is some minor intersection of our development with that of [6]. The case $f \neq 0$ seems genuinely more complex than the case $f = 0$, and our main point is not only Theorem 1.2 but its proof, which is of independent interest.

To simplify notation, let $x_j, y_k, f_j, g_k \in X$ and $\gamma_j, \delta_k > 0$ be given for $j = 0, 1, \cdots, M$ and $k = 0, 1, \cdots, N$. Assume that

$$\begin{cases} \dfrac{x_j - x_{j-1}}{\gamma_j} + Ax_j \ni f_j & \text{for } j = 1, \cdots, M, \\[2ex] \dfrac{y_k - y_{k-1}}{\delta_k} + Ay_k \ni g_k & \text{for } k = 1, \cdots, N. \end{cases} \tag{1.3}$$

We seek to estimate $\| x_j - y_k \|$. If $a_{j,k} = \| x_j - y_k \|$, this problem reduces to studying recursive inequalities for the $a_{j,k}$. These inequalities are obtained in the next lemma. A preliminary definition is required first.

DEFINITION 1.4. If $x, y \in X$ and $\lambda \in \mathbf{R}, \lambda \neq 0$, then

$$[x, y]_\lambda = \lambda^{-1}(\|x + \lambda y\| - \|x\|).$$

Also

$$[x, y]_+ = \lim_{\lambda \downarrow 0}[x, y]_\lambda = \inf_{\lambda > 0}[x, y]_\lambda \tag{1.5}$$

and

$$[x, y]_- = \lim_{\lambda \uparrow 0}[x, y]_\lambda = \sup_{\lambda < 0}[x, y]_\lambda. \tag{1.6}$$

REMARK. The convexity of $\lambda \to \|x + \lambda y\|$ implies that $[x, y]_\lambda$ is nondecreasing in λ, and this gives rise to the right-hand equalities of (1.5) and (1.6).

LEMMA 1.7. *Let $x, \bar{x}, y, \bar{y}, f, g \in X$ and $\gamma, \delta > 0$. Let A be accretive and*

$$\frac{(x - \bar{x})}{\gamma} + Ax \ni f, \quad \frac{(y - \bar{y})}{\delta} + Ay \ni g. \tag{1.8}$$

Then

(a) $$\|x - y\| \leqq \frac{\delta}{\gamma + \delta}\|\bar{x} - y\| + \frac{\gamma}{\gamma + \delta}\|x - \bar{y}\| + \frac{\gamma\delta}{\gamma + \delta}[x - y, f - g]_+.$$

Moreover, if $\gamma \geqq \delta$, then

(b) $$\|x - y\| \leqq \frac{\delta}{\gamma}\|\bar{x} - \bar{y}\| + \frac{\gamma - \delta}{\gamma}\|x - \bar{y}\| + \delta[x - y, f - g]_+.$$

PROOF OF LEMMA 1.7. We recall that A is accretive exactly when $y_i \in Ax_i$, $i = 1, 2$, implies $[x_1 - x_2, y_1 - y_2]_+ \geqq 0$. Hence, by (1.8),

$$\begin{aligned} 0 &\leqq \left[x - y, \left(f + \frac{\bar{x} - x}{\gamma}\right) - \left(g + \frac{\bar{y} - y}{\delta}\right)\right]_+ \\ &\leqq [x - y, f - g]_+ + \left[x - y, \frac{\bar{x} - x}{\gamma}\right]_+ + \left[x - y, \frac{y - \bar{y}}{\delta}\right]_+ \\ &\leqq [x - y, f - g]_+ + \frac{1}{\gamma}(\|\bar{x} - y\| - \|x - y\|) + \frac{1}{\delta}(\|x - \bar{y}\| - \|x - y\|). \end{aligned}$$

Rearranging gives (a). (We used here the facts that $[x, y + z]_+ \leqq [x, y]_+ + [x, z]_+$ and $[x, y]_+ \leqq [x, y]_\lambda$ for $\lambda > 0$ and $x, y, z \in X$.) Part (b) is obtained in a similar way.

LEMMA 1.9. *If* (1.3) *holds and $a_{j,k} = \|x_j - y_k\|$, then*

$$a_{j,k} \leqq \frac{\delta_k}{\gamma_j + \delta_k}a_{j-1,k} + \frac{\gamma_j}{\gamma_j + \delta_k}a_{j,k-1} + \frac{\delta_k\gamma_j}{\gamma_j + \delta_k}h_{j,k} \tag{1.10}$$

for $h_{j,k} = [x_j - y_k, f_j - g_k]_+$ or $h_{j,k} = \|f_j - g_k\|$.

Lemma 1.9 follows at once from Lemma 1.7(a) and the estimate$|[x-y, f-g]_+| \leqq \|f-g\|$. The task then is to estimate solutions of (1.10). This is done in the next section.

2. The main estimates

Given partitions $0 = s_0 < s_1 < \cdots < s_M = S$ and $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ of $[0, S]$ and $[0, T]$ we form the grid $\Delta = \{(s_j, \tau_k): j = 1, \cdots, M$ and $k = 1, \cdots, N\}$ on $\Omega = (0, S] \times (0, T]$. The quantities $\gamma_j = s_j - s_{j-1}$, $\delta_k = \tau_k - \tau_{k-1}$, $\mu = \max\{\gamma_j, \delta_k: j = 1, \cdots, M; k = 1, \cdots, N\}$, S, T and the sets $\Omega, \bar{\Omega} = [0, S] \times [0, T]$ and $\partial\Omega = \bar{\Omega} \setminus \Omega$ are all regarded as functions of Δ. Since Δ will be fixed for some time this dependence is not indicated explicitly now. Below j and k are understood to assume the values $j = 1, \cdots, M$ and $k = 1, \cdots, N$ unless $j = 0$ or $k = 0$ is stated explicitly. If $h: \Omega \to \mathbf{R}$, then $h_{j,k} = h(s_j, \tau_k)$ and $h_\Delta: \Omega \to \mathbf{R}$ is defined by $h_\Delta(s, \tau) = h_{j,k}$ on $(s_{j-1}, s_j] \times (\tau_{k-1}, \tau_k]$. If $h = h_\Delta$, h is said to be piecewise constant on Δ. Let $u: \Omega \to \mathbf{R}$ satisfy $u_s + u_\tau = h(s, \tau)$ on Ω. If the mesh μ is small enough and u is smooth enough, then it is easy to see that the piecewise constant function e on Δ defined by

$$\frac{u_{j,k} - u_{j-1,k}}{\gamma_j} + \frac{u_{j,k} - u_{j,k-1}}{\delta_k} = e_{j,k} + h_{j,k} \tag{2.1}$$

is small. This is made precise below. Solving (2.1) for $u_{j,k}$ leads to

$$u_{j,k} = \frac{\delta_k}{\gamma_j + \delta_k} u_{j-1,k} + \frac{\gamma_j}{\gamma_j + \delta_k} u_{j,k-1} + \frac{\gamma_j \delta_k}{\gamma_j + \delta_k} (h_{j,k} + e_{j,k}) \tag{2.2}$$

and the relationship with (1.10) is obvious.

LEMMA 2.3. *Let $u_s, u_\tau \in C(\Omega)$, $u_s + u_\tau = h$ on Ω and $u_{ss}, u_{\tau\tau} \in L^\infty(\Omega)$. If $e = e_\Delta$ is defined by* (2.1) *or* (2.2), *then*

$$| e_{j,k} | \leqq \gamma_j \| u_{ss} \|_{L^\infty} + \delta_k \| u_{\tau\tau} \|_{L^\infty}.$$

PROOF OF LEMMA 2.3. By assumption, $u_{sj,k} + u_{\tau j,k} = h_{j,k}$. On the other hand

$$\left| u_{sj,k} - \frac{u_{j,k} - u_{j-1,k}}{\gamma_j} \right| \leqq \gamma_j \| u_{ss} \|_{L^\infty}$$

and

$$\left| u_{\tau j,k} - \frac{u_{j,k} - u_{j,k-1}}{\delta_k} \right| \leqq \delta_k \| u_{\tau\tau} \|_{L^\infty}$$

follow from elementary considerations and the result is established.

DEFINITION 2.4. Let $\omega: [-S, T] \to \mathbf{R}$ and $h: \Omega \to \mathbf{R}$. Then $G(\omega, h): \Omega \to \mathbf{R}$ is given by

$$(2.5)\quad G(\omega,h)(s,\tau) = \omega(\tau - s) + \begin{cases} \displaystyle\int_0^s h(\alpha, \tau - s + \alpha)\, d\alpha & \text{if} \quad \tau \geqq s \\ \displaystyle\int_0^\tau h(s - \tau + \alpha, \alpha)\, d\alpha & \text{if} \quad s \geqq \tau. \end{cases}$$

Also, $H(\omega, h)$ is the piecewise constant function on Δ defined by $H(\omega, h)_{j,k} = b_{j,k}$ where

$$(2.6)\quad \begin{aligned} &\text{(a)} \quad b_{j,k} = \frac{\delta_k}{\gamma_j + \delta_k} b_{j-1,k} + \frac{\gamma_j}{\gamma_j + \delta_k} b_{j,k-1} + \frac{\delta_k \gamma_j}{\delta_k + \gamma_j} h_{j,k}, \\ &\text{(b)} \quad b_{j,k} = \omega(\tau_k - s_j) \quad \text{if} \quad j = 0 \quad \text{or} \quad k = 0. \end{aligned}$$

Informally, $u = G(\omega, h)$ is the solution of $u_s + u_\tau = h$ on $\Omega, u(s,\tau) = \omega(\tau - s)$ on $\partial\Omega$. We have not been precise about regularity requirements on ω and h. Below, ω will be continuous and in (2.5) the indicated integrals must be defined. According to (2.2) and Lemma 2.3, $u = G(\omega, h)$ satisfies $u_{j,k} = H(\omega, h + e)_{j,k}$ where e is small if u is smooth and the mesh μ is small. The main result of this section is an estimate of $H(\omega, h) - G(\omega, h)$. Some definitions are needed first.

DEFINITION 2.7. Let $h: \Omega \to \mathbf{R}$. Then

(a) $\| h \|^* = \inf\{\| g \|_{L^1} + \| f \|_{L^1}: g \in L^1(0, S),\ f \in L^1(0, T)$ and $| h(s,\tau)| \leqq g(s) + f(\tau)$ a.e. on $\Omega\}$.

(b) If $h \in C(\bar{\Omega})$ then $\| h \|^{**} = \| G(0, | h |)\|_{L^\infty(\Omega)}$.

Y denotes the completion of $C(\bar{\Omega})$ under $\| \ \|^{**}$ and W is the completion of $C(\bar{\Omega})$ under $\| \ \|^*$.

REMARKS. $\| \ \|^{**}$ is a weakest norm for which the corresponding completion Y has the property that G extends to a continuous linear mapping G: $C([-S, T]) \times Y \to C(\bar{\Omega})$. A calculation shows immediately that $\| \ \|^{**} \leqq \| \ \|^*$ on $C(\bar{\Omega})$; so $W \subset Y$ with a continuous injection, and $G: C([-S, T]) \times W \to C(\bar{\Omega})$ is defined and continuous. Moreover the $L^1(\Omega)$ norm is weaker than $\| \ \|^{**}$, as is easy to see, and $\| h \|^* \leqq \min(S, T) \| h \|_{L^\infty}$ for $h \in C(\bar{\Omega})$. Thus,

$$C(\bar{\Omega}) \subset W \subset Y \subset L^1(\Omega)$$

where each injection is continuous and has dense range. More precise characterizations of Y and W need not concern us here.

The main estimate is

THEOREM 2.8. *Let $\omega \in C([-S, T])$, and $h:\Omega \to \mathbf{R}$ be piecewise constant on Δ. If $\tilde{\omega} \in C^2([-S, T])$, $\tilde{h} \in C^2(\bar{\Omega})$ and $\tilde{h}(0,0) = 0$ then*

$$\| H(\omega, h) - G(\omega, h) \|_{L^\infty(\Omega)} \leqq 2 \| \omega - \tilde{\omega} \|_{L^\infty(-S,T)} + 2 \| h - \tilde{h} \|^* + \| \tilde{h} - \tilde{h}_\Delta \|^*$$
$$+ \mu((T+S)\| \tilde{\omega}'' \|_{L^\infty(-S,T)} + 2 \| \tilde{\omega}' \|_{L^\infty(-S,T)}$$
$$+ 2 \| \tilde{h} \|_{C^2(\Omega)} (1 + T + S)^2).$$

The proof of Theorem 2.8 is postponed while some consequences, including Theorem 1.2, are obtained. The precise coefficient of μ above is not of much interest and is not optimal. Its nature will be clear when the proof is given.

Let I be a partially ordered set and $\{\Delta(i): i \in I\}$ be a net of grids. S_i, T_i, Ω_i, μ_i, etc., and operators H_i are associated with $\Delta(i)$, as S, T, μ, Ω were associated with Δ above. The norm $\| \; \|^*$ and space W over Ω_i will be denoted by $\| \; \|_i^*$ and W_i. Let also $S_0, T_0 > 0$ and $\Omega_0 = (0, S_0] \times (0, T_0]$ be given.

COROLLARY 2.9. *Let $S_0 \geqq S_i$, $T_0 \geqq T_i$ and $\omega_i \in C([-S_i, T_i])$ for $i \in I$. Let $h_i: \Omega_i \to \mathbf{R}$ be piecewise constant over Δ_i. If $h \in W_0$, $\omega \in C([-S_0, T_0])$ and*

$$\lim_I \mu_i = \lim_I \| h_i - h \|_i^* = \lim_I \| \omega - \omega_i \|_{L^\infty(-S_i, T_i)} = 0$$

then

$$\lim_I \| H_i(\omega_i, h_i) - G(\omega, h) \|_{L^\infty(\Omega_i)} = 0.$$

PROOF OF COROLLARY 2.9. By the remarks following Definition 2.7

$$\| G(\omega, h) - G(\omega_i, h_i) \|_{L^\infty(\Omega_i)} = \| G(\omega - \omega_i, h - h_i) \|_{L^\infty(\Omega_i)}$$
$$\leqq \| \omega - \omega_i \|_{L^\infty(-S_i, T_i)} + \| h - h_i \|_i^*$$

and the right hand side tends to zero by assumption. Thus, it is enough to show that $\| H_i(\omega_i, h_i) - G(\omega_i, h_i) \|_{L^\infty(\Omega_i)}$ tends to zero. Theorem 2.8 and the triangle inequality imply that

$$\| H_i(\omega_i, h_i) - G(\omega_i, h_i) \|_{L^\infty(\Omega_i)} \leqq 2 \| \omega - \omega_i \|_{L^\infty(-S_i, T_i)}$$
$$+ 2 \| \omega - \tilde{\omega} \|_{L^\infty(-S_0, T_0)} + 2 \| h - h_i \|_i^* + 2 \| h - \tilde{h} \|_0^*$$
$$+ \| \tilde{h} - \tilde{h}_{\Delta_i} \|_i^* + \mu_i K(\| \tilde{\omega} \|_{C^2([-S_0, T_0])} + \| \tilde{h} \|_{C^2(\bar{\Omega}_0)})$$

where K is independent of i, provided that $\tilde{\omega} \in C^2([-S_0, T_0])$, $\tilde{h} \in C^2(\bar{\Omega}_0)$ and $\tilde{h}(0,0) = 0$. The limit over I of every term on the right hand side above in which an i appears is zero. For the term $\| \tilde{h} - \tilde{h}_{\Delta_i} \|_i^*$ this follows from $\| \; \|_i^* \leqq \min(S_i, T_i) \| \; \|_{L^\infty(\Omega_i)}$, the uniform continuity of $\tilde{h}$, and $\mu_i \to 0$. The other terms tend to zero by assumption. Thus

$$\limsup_I \| H_i(\omega_i, h_i) - G(\omega_i, h_i) \|_{L^\infty(\Omega_i)} \leqq 2 \| \omega - \tilde{\omega} \|_{L^\infty([-S_0, T_0])} + 2 \| h - \tilde{h} \|_0^*.$$

Now $\tilde{\omega} \in C^2([-S_0, T_0])$ and $\tilde{h} \in C^2(\bar{\Omega}_0)$ with $\tilde{h}(0,0) = 0$ can be chosen so the right-hand side is as small as desired, and the result is proved. (The density of such $(\tilde{\omega}, \tilde{h})$ in $C([-S_0, T_0]) \times W_0$ is an exercise.)

Proof of Theorem 1.2. For $m, n > 0$ define

$$\Delta_{m,n} = \{(t_j^m, t_k^n) : j = 1, \cdots, N(m);\ k = 1, \cdots, N(n)\}.$$

The functions $w^{m,n}(s,\tau) = \| u_m(s) - u_n(\tau) \|$, $h^{m,n}(s,\tau) = \| f_m(s) - f_n(\tau) \|$ are piecewise constant on the grid $\Delta_{m,n}$ on $\Omega_{m,n} = (0, T(m)] \times (0, T(n)]$. Now $H_{m,n}(\omega, h) \geqq 0$ if $\omega \geqq 0$ and $h \geqq 0$. It follows from this and Lemma 1.9 that if

$$w^{m,n}(t_j^m, t_k^n) = \| x_j^m - x_k^n \| \leqq \omega^{m,n}(t_k^n - t_j^m) \quad \text{for} \quad j = 0 \quad \text{or} \quad k = 0, \tag{2.10}$$

then

$$w^{m,n}(s,\tau) = \| u_m(s) - u_n(\tau) \| \leqq H_{m,n}(\omega^{m,n}, h^{m,n})(s,\tau). \tag{2.11}$$

Analogously to Kobayashi [6], if we take

$$\begin{cases} \omega^{m,n}(\tau - s) = \displaystyle\int_0^{|\tau - s|} (\| f(\alpha) \| + \| y \|)\, d\alpha + 2(\| x_0^m - x \| + \| x_0^n - x \|) \\ \qquad\qquad + \| f - f_m \|_{L^1(0,T(m):X)} + \| f - f_n \|_{L^1(0,T(n):X)} \\ \text{where} \quad x \in D(A) \quad \text{and} \quad y \in Ax, \end{cases} \tag{2.12}$$

then (2.10) is satisfied. This follows from a simple induction once we notice that for $y \in Ax$

$$\begin{aligned} \| x_k^n - x \| &\leqq \left\| x_k^n - x + (t_k^n - t_{k-1}^n)\left(f_k^n + \frac{x_{k-1}^n - x_k^n}{t_k^n - t_{k-1}^n} - y \right) \right\| \\ &\leqq \| x_{k-1}^n - x \| + (t_k^n - t_{k-1}^n)(\| f_k^n \| + \| y \|) \\ &\leqq \| x_{k-1}^n - x \| + \int_{t_{k-1}^n}^{t_k^n} (\| f(\alpha) \| + \| y \| + \| f(\alpha) - f_n(\alpha) \|)\, d\alpha \end{aligned}$$

where the first inequality is due to the accretiveness of A. Thus

$$\|x_k^n - x_0^n\| \leqq \|x_k^n - x\| + \|x - x_0^n\|$$

$$\leqq \int_0^{t_k^n} (\|f(\alpha)\| + \|y\|)\, d\alpha + 2\|x_0^n - x\| + \|f - f_n\|_{L^1(0,T(n):X)}.$$

Next observe that

$$|h^{m,n}(s,\tau) - \|f(s) - f(\tau)\|| \leqq \|f_m(s) - f(s)\| + \|f_n(\tau) - f(\tau)\|.$$

Thus $\|h^{m,n} - \|f(s) - f(\tau)\|\|^*_{m,n} \to 0$ as $m, n \to \infty$. Applying Corollary 2.9 to (2.11) then yields

$$\limsup_{m,n\to\infty} \|u_m(s) - u_n(\tau)\| \leqq \limsup_{m,n\to\infty} H_{m,n}(\omega^{m,n}, h^{m,n})(s,\tau) = G(\omega, h)(s,\tau)$$

uniformly on $0 \leqq s, \tau \leqq T$ where

$$\omega(\tau - s) = \int_0^{|\tau - s|} (\|f(\alpha)\| + \|y\|)\, d\alpha + 4\|x_0 - x\|$$

and

$$h(s,\tau) = \|f(s) - f(\tau)\|.$$

Since $G(\omega, h)(t,t) = 4\|x_0 - x\|$, $x_0 \in \overline{D(A)}$ and $x \in D(A)$ is arbitrary, it follows that $\|u_m(t) - u_n(t)\| \to 0$ uniformly on $0 \leqq t \leqq T$ and $u = \lim u_n$ thus exists uniformly on $[0, T]$. Then if $0 \leqq s < \tau \leqq T$ we also find

$$\|u(s) - u(\tau)\| \leqq \int_0^{\tau - s} (\|f(\alpha)\| + \|y\|)\, d\alpha + 4\|x_0 - x\|$$

$$+ \int_0^s \|f(\tau - s + \alpha) - f(\alpha)\|\, d\alpha$$

for $y \in Ax$. The continuity of u follows easily. Moreover, one sees how the modulus of continuity of u may be estimated in terms of f and $\inf\{(\tau - s)\|y\| + 4\|x_0 - x\| : x \in D(A), y \in Ax\}$. (One may replace 4 by 2 in this expression.) For example, if $x_0 \in D(A)$ and f is of bounded variation, then u is Lipschitz continuous.

Proof of Theorem 2.8. The main ingredients in the proof are the next two lemmas.

Lemma 2.13. *Let* $\omega \in C([-S, T])$ *and* $h : \Omega \to \mathbf{R}$. *Then* $\|H(\omega, h)\|_{L^\infty} \leqq \|\omega\|_{L^\infty} + \|h_\Delta\|^*$.

PROOF OF LEMMA 2.13. First, $H(\omega,h)=H(\omega,0)+H(0,h)$ and, by the definition of H, the values of $H(\omega,0)$ are convex combinations of the values of ω. Thus $\| H(\omega,0)\|_{L^\infty} \leqq \|\omega\|_{L^\infty}$. It remains to see that $\|H(0,h)\|_{L^\infty} \leqq \|h_\Delta\|^*$. Now

$$\|h_\Delta\|^* = \inf\{\gamma_1\beta_1 + \cdots + \gamma_M\beta_M + \delta_1\kappa_1 + \cdots + \delta_N\kappa_N : \tag{2.14}$$
$$|h_{j,k}| \leqq \beta_j + \kappa_k \quad \text{and} \quad \beta_j, \kappa_k \geqq 0\},$$

as is easy to see. Now let $g_{j,k} = \beta_j + \kappa_k \geqq |h_{j,k}|$ and

$$b_{j,k} = \gamma_1\beta_1 + \cdots + \gamma_j\beta_j + \delta_1\kappa_1 + \cdots + \delta_k\kappa_k.$$

Then $b = H(\omega, g)$ provided $\omega(\tau_k - s_j) = b_{j,k}$ for $j=0$ or $k=0$; that is, b satisfies (2.6) (a) with $h_{j,k}$ replaced by $g_{j,k}$. Since $g_{j,k} \geqq |h_{j,k}|$, and $\beta_j, \kappa_k \geqq 0$ (so we may take $\omega \geqq 0$), we have $b = H(\omega,g) \geqq H(0,|h|) \geqq H|(0,h)|$ by the order preserving property of H. Thus,

$$\|H(0,h)\|_{L^\infty} \leqq \gamma_1\beta_1 + \cdots + \gamma_M\beta_M + \delta_1\kappa_1 + \cdots + \delta_N\kappa_N$$

and, in view of (2.14), the proof is complete.

LEMMA 2.15. *Let $\omega \in C([-S,T])$, $h \in C(\Omega)$ and $u = G(\omega,h)$ satisfy the conditions of Lemma* 2.3. *Then*

$$\|H(\omega,h) - G(\omega,h)\|_{L^\infty} \leqq \mu(T\|u_{\tau\tau}\|_{L^\infty} + S\|u_{ss}\|_{L^\infty} + \|u_s\|_{L^\infty} + \|u_\tau\|_{L^\infty}).$$

PROOF OF LEMMA 2.15. Let e be the piecewise constant function on Δ defined by (2.2). Then, by definition,

$$H(\omega,h) - G(\omega,h)_\Delta = H(0,e). \tag{2.16}$$

On the other hand, by Lemma 2.3 and Lemma 2.13,

$$\begin{aligned}\|H(0,e)\|_{L^\infty} &\leqq \|e\|^* \\ &\leqq (\gamma_1^2 + \cdots + \gamma_M^2)\|u_{ss}\|_{L^\infty} \\ &\quad + (\delta_1^2 + \cdots + \delta_N^2)\|u_{\tau\tau}\|_{L^\infty} \\ &\leqq \mu(S\|u_{ss}\|_{L^\infty} + T\|u_{\tau\tau}\|_{L^\infty}).\end{aligned} \tag{2.17}$$

Furthermore we have

$$\|G(\omega,h)_\Delta - G(\omega,h)\|_{L^\infty} \leqq \mu(\|u_s\|_{L^\infty} + \|u_\tau\|_{L^\infty}) \tag{2.18}$$

since $\|u_s\|_{L^\infty} + \|u_\tau\|_{L^\infty}$ is a Lipschitz constant for $u = G(\omega,h)$. Combining (2.16)–(2.18) yields the lemma.

END OF PROOF OF THEOREM 2.8. Let $\tilde{\omega} \in C([-S, T])$, $\tilde{h} \in C(\Omega)$ and $u = G(\tilde{\omega}, \tilde{h})$ satisfy the assumptions of Lemma 2.3. Then

$$\begin{aligned}
\| H(\omega, h) - G(\omega, h)\|_{L^\infty} &\leqq \| H(\omega - \tilde{\omega}, h - \tilde{h})\|_{L^\infty} + \| G(\omega - \tilde{\omega}, h - \tilde{h})\|_{L^\infty} \\
&\quad + \| H(\tilde{\omega}, \tilde{h}) - G(\tilde{\omega}, \tilde{h})\|_{L^\infty} \\
&\leqq 2\| \omega - \tilde{\omega} \|_{L^\infty} + \|(h - \tilde{h})_\Delta\|^* + \| h - \tilde{h} \|^* + \| H(\tilde{\omega}, \tilde{h}) - G(\tilde{\omega}, \tilde{h})\|_{L^\infty} \\
&\leqq 2\| \omega - \tilde{\omega} \|_{L^\infty} + 2\| h - \tilde{h} \|^* + \| \tilde{h} - \tilde{h}_\Delta \|^* \\
&\quad + \mu(S \| u_{ss} \|_{L^\infty} + T \| u_{\tau\tau} \|_{L^\infty} + \| u_\tau \|_{L^\infty} + \| u_s \|_{L^\infty})
\end{aligned}$$

by Lemmas 2.13 and 2.15. If $\tilde{\omega} \in C^2([-S, T])$, $\tilde{h} \in C^2(\bar{\Omega})$ and $\tilde{h}(0,0) = 0$, then $u = G(\tilde{\omega}, \tilde{h})$ has the required regularity, and the result follows from examining the dependence of the derivatives of u on $\tilde{\omega}$ and $\tilde{h}$, which task we leave to the reader. The theorem is proved.

3. Remarks on Benilan's uniqueness theorem

According to Benilan [1], if A is accretive, $\mathcal{I}$ is an interval, and $g \in L^1_{\text{loc}}(\mathcal{I}: X)$, then v is an *integral solution* of $v' + Av \ni g$ on $\mathcal{I}$ provided that $v \in C(\mathcal{I}: X)$ and

$$\text{(3.1)} \quad \begin{cases} \| v(t) - x \| - \| v(s) - x \| \leqq \int_s^t [v(\tau) - x, g(\tau) - y]_+ d\tau \\ \text{whenever } x \in D(A),\ y \in Ax,\ s, t \in \mathcal{I} \text{ and } s \leqq t. \end{cases}$$

The following result was proved in [1].

THEOREM 3.2. (Benilan) *Let the assumptions of Theorem 1.2 be satisfied and $u = \lim_{n\to\infty} u_n$ with the notation of Theorem 1.2. Let $g \in L^1(0, T: X)$. If v is an integral solution of $v' + Av \ni g$ on $[0, T]$, then*

$$\text{(3.3)} \quad \| v(t) - u(t)\| - \| v(s) - u(s)\| \leqq \int_s^t [v(\tau) - u(\tau), g(\tau) - f(\tau)]_+ d\tau$$

for $0 \leqq s \leqq t \leqq T$.

We sketch a proof of this result which, while similar to Benilan's, exhibits the nature of the situation more clearly. First observe that (3.1) is equivalent to

$$\text{(3.1')} \quad \frac{d}{dt}\| v(t) - x \| \leqq [v(t) - x, g(t) - y]_+ \quad \text{in} \quad \mathcal{D}'((0, T))$$

for $y \in Ax$. Setting

$$x = x_k^n,\ y = f_k^n + \frac{x_{k-1}^n - x_k^n}{\delta_k^n},\ \delta_k^n = t_k^n - t_{k-1}^n$$

in (3.1′) and using that

$$[x, z+w]_+ \leqq [x, z]_\lambda + [x, w]_\delta$$

for all $x, y, w \in X$ and $\lambda, \delta > 0$ leads to

$$(3.4)\ \frac{d}{dt}\| v(t) - x_k^n \| \leqq [v(t) - x_k^n, g(t) - f_k^n]_\lambda + [v(t) - x_k^n, (\delta_k^n)^{-1}(x_k^n - x_{k-1}^n)]_{\delta_k^n}$$

$$= [v(t) - x_k^n, g(t) - f_k^n]_\lambda + \frac{\| v(t) - x_{k-1}^n \| - \| v(t) - x_k^n \|}{\delta_k^n}$$

for $\lambda > 0$. Let

$$(3.5)\qquad \begin{cases} w_n(t,s) = \| v(t) - u_n(s) \|, \\ h_n(t,s) = [v(t) - u_n(s), g(t) - f_n(s)]_\lambda, \\ g_n(t,s) = \dfrac{\| v(t) - x_k^n \| - \| v(t) - x_{k-1}^n \|}{\delta_k^n},\ t_{k-1}^n < s \leqq t_k^n. \end{cases}$$

Now (3.4) may be restated as

$$(3.6)\qquad \frac{\partial}{\partial t} w_n(t,s) + g_n(t,s) \leqq h_n(t,s) \quad \text{in} \quad \mathcal{D}'((0,T)\times(0,T)).$$

We have $w_n(t,s) \to \| v(t) - u(s) \|$ uniformly, and

$$h_n(t,s) \to [v(t) - u(s), g(t) - f(s)]_\lambda \quad \text{in} \quad W((0,T)\times(0,T))$$

(W is defined in Definition 2.7). To evaluate $\lim_n g_n(t,s)$ we use the next lemma.

LEMMA 3.7. *Let $\Omega \subseteq \mathbf{R}^N$ be open and $\psi: \Omega \times X \to \mathbf{R}$ be continuous. Let $T > 0$, $0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n = T$ be a sequence of partitions of $[0, T]$, and $\{u_n\}$ be a sequence of strongly measurable functions on $[0, T]$ such that $\lim_{n\to\infty} u_n = u \in C([0,T]: X)$ holds uniformly on $[0,T]$. If $\lim_{n\to\infty} \max_{1\leqq k \leqq N(n)} (t_k^n - t_{k-1}^n) = 0$ and*

$$g_n(z,s) = \frac{\psi(z, u_n(t_k^n)) - \psi(z, u_n(t_{k-1}^n))}{t_k^n - t_{k-1}^n} \quad \text{for} \quad z \in \Omega,\ t_{k-1}^n < s \leqq t_k^n,$$

then $\lim_n g_n(z,s) = (\partial/\partial s)\psi(z, u(s))$ in $\mathcal{D}'(\Omega \times (0,T))$.

PROOF OF LEMMA 3.7. Let us first assume that $\psi(z,x)=\psi(x)$ depends only on $x\in X$. Now $v_n=\psi(u_n)$ converges uniformly to the continuous function $v=\psi(u)$, and $g_n(s)=(v_n(t_k^n)-v_n(t_{k-1}^n))/(t_k^n-t_{k-1}^n)$ on $(t_{k-1}^n,t_k^n]$. Next observe that g_n is the derivative of the function $w_n(s)$ which is linear on $(t_{k-1}^n,t_k^n]$ and has the value $v_n(t_k^n)$ at t_k^n. Clearly $\|w_n\|_{L^\infty}\leqq\|v_n\|_{L^\infty}$ and $w_n\to v=\psi(u)$ uniformly. Since $g_n=w_n'$, it follows that $g_n\to v'$ in $\mathcal{D}'((0,T))$, and we have the result. Moreover, observe that

$$\text{(3.8)}\qquad \left|\int_0^T g_n(s)\phi(s)ds\right|=\left|\int_0^T w_n'(s)\phi(s)ds\right|=\left|\int_0^T w_n(s)\phi'(s)ds\right|$$
$$\leqq\|w_n\|_{L^\infty}\|\phi'\|_{L^1}=\|\psi(u_n)\|_{L^\infty}\|\phi'\|_{L^1},$$

for $\phi\in\mathcal{D}'((0,T))$. To obtain the general result, observe that

$$\lim_n\int_\Omega\left(\int_0^T g_n(z,s)\phi(z,s)ds\right)dz=\int_\Omega\left(\lim_n\int_0^T g_n(z,s)\phi(z,s)ds\right)dz$$
$$=-\int_\Omega\left(\int_0^T\psi(z,u(s))\frac{\partial}{\partial s}\phi(z,s)ds\right)dz$$

for $\phi\in\mathcal{D}(\Omega\times(0,T))$ by the case treated above and (3.8) (which allows the interchange of the integral over Ω and the limit on n).

To complete the proof of the theorem, we use Lemma 3.7 to pass to the limit in (3.6) to find

$$\text{(3.9)}\qquad \frac{\partial}{\partial t}\|v(t)-u(s)\|+\frac{\partial}{\partial s}\|v(t)-u(s)\|\leqq[v(t)-u(s),g(t)-f(s)]_\lambda$$

in $\mathcal{D}'((0,T)\times(0,T))$. It is only an exercise to integrate the inequality (3.9) and let $\lambda\downarrow 0$ to find (3.3).

REMARK. The reader may recognize the relationship of Lemma 3.7 to the general question of when $u_n\to u$ uniformly and $\delta_n>0$, $\delta_n\to 0$ uniformly implies $(u_n(t)-u_n(t-\delta_n(t)))/\delta_n(t)\to u'(t)$ in $\mathcal{D}((0,T))$. Even if $u_n=u$ is independent of n and u is absolutely continuous, this does not hold in general.

REMARK. If A is accretive and u, v are strong solutions of $u'+Au\ni f$, $v'+Av\ni g$, one finds (3.3) directly. If A is quasi-accretive in the sense of Takahashi [10], one finds instead

$$\text{(3.10)}\qquad \begin{aligned}&\frac{\partial}{\partial t}\|v(t)-u(s)\|+\frac{\partial}{\partial s}\|v(t)-u(s)\|\\&\leqq[v(t)-u(s),g(t)]_++[v(t)-u(s),-f(s)]_+.\end{aligned}$$

This suggests the appropriate notion of integral solution in this case, as well as the (verifiable) version of Theorem 3.2. Similarly, one can discuss the case where $A+\nu I$ is accretive for some $\nu \in \mathbf{R}$.

4. Remarks, extensions and special cases

REMARK 4.1. (On Lemma 1.7). If A is only quasi-accretive in the sense of Takahashi [10], then Lemma 1.7(a) remains valid if $[x-y,f-g]_+$ is replaced by $[x-y,f]_+ + [x-y,-g]_+ \leqq \|f\| + \|g\|$. Lemma 1.7(b), however, holds only when A is accretive. Kobayashi [6] also proved Lemma 1.7(a) in the form valid for quasi-accretive operators. In any case, this inequality is suggested by Takahashi [10] who stated the result for $\gamma=\delta$. If $\nu \in \mathbf{R}$ and $A+\nu I$ is accretive, writing (1.8) as $\gamma^{-1}(x-\bar{x})+Ax+\nu x \ni f+\nu x$, etc., and using Lemma 1.7 for $(A+\nu I)$ yields

$$\left(1-\frac{\nu\gamma\delta}{\gamma+\delta}\right)\|x-y\| \leqq \frac{\delta}{\gamma+\delta}\|\bar{x}-y\| + \frac{\gamma}{\gamma+\delta}\|x-\bar{y}\| + \frac{\gamma\delta}{\gamma+\delta}[x-y,f-g]_+. \tag{4.2}$$

REMARK 4.3. (On Lemma 1.9). If A is only quasi-accretive or $A+\nu I$ is accretive, then Lemma 1.9 changes in accordance with Remark 4.1.

REMARK 4.4. (On schemes for $u_s+u_\tau=h$). One may ask for what coefficients does the scheme

$$u_{j,k}=\theta_{j,k}u_{j-1,k}+\beta_{j,k}u_{j,k-1}+\eta_{j,k}u_{j-1,k-1}+\kappa_{j,k}h_{j,k} \tag{4.5}$$

represent a difference approximation for $u_s+u_\tau=h$. If $\theta_{j,k},\beta_{j,k},\eta_{j,k}\geqq 0$ and $\theta_{j,k}+\beta_{j,k}+\eta_{j,k}=1$, the conditions are $(\theta_{j,k}+\eta_{j,k})\gamma_j=(\beta_{j,k}+\eta_{j,k})\delta_k=\kappa_{j,k}$. One solution of this is $\eta_{j,k}=0$, $\theta_{j,k}=\delta_k/(\gamma_j+\delta_k)$, $\beta_{j,k}=\gamma_j/(\gamma_j+\delta_k)$, $\kappa_{j,k}=\gamma_j\delta_k/(\gamma_j+\delta_k)$. This corresponds to the development in Section 2. If $\gamma_j>\delta_k$, a second solution is $\eta_{j,k}=\delta_k/\gamma_j$, $\theta_{j,k}=0$, $\beta_{j,k}=(\gamma_j-\delta_k)/\gamma_j$, $\kappa_{j,k}=\delta_k$ which corresponds to Lemma 1.7(b) in the same sense that the previous scheme corresponds to Lemma 1.7(a). The other possibilities are convex combinations of these extreme ones. It is curious that the nonsymmetric extreme Lemma 1.7(b), used in [4], was discovered before the symmetric extreme. The analysis of Section 2 can be adapted to the general case (4.5).

REMARK 4.6. (On Theorem 1.2 if $A+\nu I$ is accretive). Theorem 1.2 remains valid if $A+\nu I$ is accretive for some $\nu \in \mathbf{R}$. One way to prove this would be to treat the inequalities

$$\left(1-\frac{\nu\gamma_j\delta_k}{\gamma_j+\delta_k}\right)a_{j,k}\leqq\frac{\delta_k}{\gamma_j+\delta_k}a_{j-1,k}+\frac{\gamma_j}{\gamma_j+\delta_k}a_{j,k-1}+\frac{\gamma_j\delta_k}{\gamma_j+\delta_k}h_{j,k}\tag{4.7}$$

via comparison with solutions of the equation $u_s+u_\tau-\nu u=h$ as was done for $\nu=0$. However, just as the νu term may be transformed out in this equation by changes of variables, a similar device works for (4.7). Set, for example,

$$b_{j,k}=(1-\nu\gamma_1)\cdots(1-\nu\gamma_j)\,a_{j,k}$$

$$g_{j,k}=(1-\nu\gamma_1)\cdots(1-\nu\gamma_j)\,h_{j,k}.$$

Then (4.7) becomes

$$\begin{aligned}&b_{j,k}\leqq\frac{\delta_k}{\gamma_j^*+\delta_k}b_{j-1,k}+\frac{\gamma_j^*}{\gamma_j^*+\delta_k}b_{j,k-1}+\frac{\gamma_j^*\delta_k}{\gamma_j^*+\delta_k}g_{j,k},\\&\gamma_j^*=\gamma_j/(1-\gamma_j\nu).\end{aligned}\tag{4.8}$$

Or, using $b_{j,k}=(1-(\nu/2)\gamma_1)\cdots(1-(\nu/2)\gamma_j)(1-(\nu/2)\delta_1)\cdots(1-(\nu/2)\delta_k)a_{j,k}$, etc., one obtains (4.8) with $\delta_k^*=\delta_k/(1-(\nu/2)\delta_k)$ in place of δ_k and $\gamma_k^*=\gamma_k/(1-(\nu/2)\gamma_k)$. One then checks that the change of partition represented by the new step sizes does not effect the convergence in L^1 assumed in Theorem 1.2. This point is discussed in more detail in Remark 4.9 of the Technical Summary Report #1541, Mathematics Research Center, University of Wisconsin-Madison (which has the same title and authors as the current paper).

REMARK 4.9. (The case $\omega(t)=K\,|\,t\,|+b$). Let $a_{j,k}$ satisfy (1.10) and $a_{j,k}\leqq K\,|\,t_k-s_j\,|+b$ if $j=0$ or $k=0$. If h is piecewise constant on Δ, then

$$a_{j,k}\leqq H(\omega,h)_{j,k}=H(\omega,0)_{j,k}+\|h\|^*\tag{4.10}$$

where $\omega(t)=K\,|\,t\,|+b$. By Theorem 2.8,

$$\|H(\omega,0)-G(\omega,0)\|_{L^\infty}\leqq2\|\omega-\tilde\omega\|_{L^\infty}+\mu((T+S)\|\tilde\omega''\|_{L^\infty}+2\|\tilde\omega'\|_{L^\infty})$$

for $\tilde\omega\in C^2([-S,T])$. Reviewing the proofs, it is enough to have $\tilde\omega''\in L^\infty([-S,T])$ and in fact

$$|\,H(\omega,0)_{j,k}-G(\omega,0)_{j,k}\,|\leqq2\|\omega-\tilde\omega\|_{L^\infty}+\mu((T+S)\|\tilde\omega''\|_{L^\infty}),\tag{4.11}$$

since the term $2\mu\,\|\tilde\omega'\|_{L^\infty}$ only arose from the variation of $G(\omega,0)$ over a rectangle of the grid (that is, from the terms $\|u_s\|_{L^\infty}$ and $\|u_\tau\|_{L^\infty}$ in Lemma 2.15). With $\omega(t)=K\,|\,t\,|+b$ and $\tilde\omega$ defined by $\tilde\omega(0)=b$ and

$$\tilde{\omega}'(t)=\begin{cases}\dfrac{K}{\lambda}t & \text{for} \quad |t|\leqq\lambda\\ K \text{ sign } t & \text{for} \quad |t|>\lambda\end{cases}$$

we have

$$2\|\omega-\tilde{\omega}\|_{L^\infty(-\infty,\infty)}+\mu(T+S)\|\tilde{\omega}''\|_{L^\infty(-\infty,\infty)}=K\left(\lambda+\frac{\mu(T+S)}{\lambda}\right)$$

provided $\lambda>0$. For $\lambda=\sqrt{\mu(T+S)}$ we obtain

$$|H(\omega,0)_{j,k}-G(\omega,0)_{j,k}|\leqq 2K\sqrt{\mu}\sqrt{T+S}. \tag{4.12}$$

Combining (4.12), (4.11) and (4.10) leads to

$$\begin{aligned} a_{M,N} &\leqq G(\omega,0)(s_M,t_N)+2K\sqrt{\mu}\sqrt{s_M+t_N}+\|h\|^* \\ &= K|s_M-t_N|+b+2K\mu\sqrt{s_M+t_N}+\|h\|^*. \end{aligned} \tag{4.13}$$

Of course, (4.13) holds with M,N replaced by j,k and $\|h\|^*$ by $\|h\|_{W((0,s_j]\times(0,t_k])}$. Thus our results are, in this special case, as sharp (except for the constants) as those obtained by other methods. (Compare, e.g., [3], [4], [6]. Of course, these simple estimates are decisive only when $f=0$ in Theorem 1.2.) One can also consider other explicit choices of ω.

REMARK 4.14. If $M=N$, $f_j=g_j$, $x_j=y_j$, $\gamma_j=\delta_j$ in (1.3), then $a_{j,k}=\|x_j-x_k\|$ in Lemma 1.9. It follows that an estimate on $a_{j,0}$, $j=1,\cdots,M$ implies an estimate on $a_{j,k}$ for all j,k. If, e.g., $a_{j,0}\leqq K|t_j|$, $t_j=\gamma_1+\cdots+\gamma_j$, we could use Remark 4.9. However, the result is not good enough, since the fact that $a_{j,j}=0$ has been ignored. To use this, observe that $a_{j,k}\leqq b_{j,k}$ for $k\geqq j\geqq 0$ where $b_{j,k}$ is defined for $M\geqq k\geqq j\geqq 0$ by

$$\begin{cases} b_{j,k}=\dfrac{\gamma_k}{\gamma_j+\gamma_k}b_{j-1,k}+\dfrac{\gamma_j}{\gamma_j+\gamma_k}b_{j,k-1}+\dfrac{\gamma_j\gamma_k}{\gamma_j+\gamma_k}g_{j,k} \\ \qquad\qquad \text{for} \quad M>k\geqq j\geqq 1, \\ b_{0,k}=\omega(t_k)\geqq a_{0,k}, \\ b_{j,j}=0 \quad \text{if} \quad M\geqq j\geqq 0, \end{cases} \tag{4.15}$$

provided that $g_{j,k}\geqq h_{j,k}=\|f_j-f_k\|$ for $M>k\geqq j\geqq 1$, $\omega\in C([0,T])$, and $\omega(0)=0$. On the other hand, if $\bar{g}$ is defined by $\bar{g}_{j,j}=0$, $\bar{g}_{k,j}=-g_{j,k}$ if $M\geqq k\geqq j\geqq 1$ and $\bar{\omega}$ denotes the odd extension of ω to $[-T,T]$, the solution of (4.15) is exactly

$b_{j,k} = H(\bar{\omega}, \bar{g})_{j,k}$. Assuming, for example, $\omega(t) = Kt$ then $\bar{\omega}(t) = Kt$ and we conclude

$$\| x_j - x_k \| \leqq H(\bar{\omega}, g)_{j,k} \leqq \| g \|_{W((0,t_j]\times(0,t_k])}$$
$$+ G(\tilde{\omega}, 0) + 2 \| \bar{\omega} - \tilde{\omega} \|_{L^\infty([-T,T])} + \mu \| \tilde{\omega}'' \|_{L^\infty([-T,T])}$$

for $\tilde{\omega} \in C^2([-T, T])$, where $g_{j,k} = \| f_j - f_k \|$, $k \geqq j$ and $g_{j,k} = -g_{k,j}$. Setting $\tilde{\omega} = Kt$, we find

$$\| x_j - x_k \| \leqq \gamma_1 \| f_1 \| + \cdots + \gamma_j \| f_j \| + \gamma_1 \| f_1 \| + \cdots + \gamma_k \| f_k \| + K(t_k - t_j) \tag{4.16}$$

for $k \geqq j$, since the terms involving $\tilde{\omega}$ vanish. This is a reflection of the fact that

$$b_{j,k} = K(t_k - t_j) + \gamma_1\beta_1 + \cdots + \gamma_j\beta_j + \gamma_1\eta_1 + \cdots + \gamma_k\eta_k$$

satisfies the recursion part of (4.15) exactly if $g_{j,k} = \beta_j + \eta_k$. It is interesting how sharp our estimates are in this case. These considerations show how the simple but important inequality [6, eq. 3] discovered by Kobayashi is naturally suggested by our methods and how to generalize to other choices of ω.

REMARK 4.17. (On existence of approximate solutions). Let $\nu \in \mathbf{R}$ and $A + \nu I$ be accretive. Set

$$S(A) = \left\{ y \in X: \liminf_{\lambda \downarrow 0} \frac{\operatorname{dist}(R(I + \lambda A), x + \lambda y)}{\lambda} = 0 \quad \text{for} \quad x \in \overline{D(A)} \right\}.$$

Then if $f \in L^1(0, T: X)$, $f(t) \in \overline{S(A)}$ a.e. on $[0, T]$, $x_0 \in \overline{D(A)}$ and $\varepsilon > 0$, there are sequences $\{x_j\}_{j=0}^m$, $\{f_j\}_{j=1}^m$, $\{\gamma_j\}_{j=1}^m$ such that (1.3) holds, $\gamma_1 + \cdots + \gamma_m = T$, $0 < \gamma_j < \varepsilon$ and

$$\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \| f_j - f(\alpha) \| \, d\alpha < \varepsilon$$

where $t_j = \gamma_1 + \cdots + \gamma_j$. Kobayashi [6] proves this if $0 \in S(A)$ and $f(t) \equiv 0$. It is not difficult to generalize his argument to the current case. It follows from Kobayashi's result and a theorem of Benilan [1] that if $A + \nu I$ is accretive then $\bar{A} + \nu I$ is m-accretive if and only if $\overline{S(A)} = X$ (our definition of $S(A)$ differs from Benilan's). It is straightforward to prove this directly by our methods, without the notion of "bonne solutions".

REMARK 4.18. L. Evans [5] has applied arguments of the type employed here to study the equation $du/dt + A(t)u \ni f(t)$ where $A(t)$ has an "L^1-modulus of continuity".

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